Rose's Modern Geometry

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Contents

1 Synthetic Geometry

1.1 Hilbert's Axioms

Synthetic geometry is axiomatic and relative. **Hilbert plane geometry** is an axiomatic system on points A, lines l, incidence $\in$ of points in lines, order $*$ on points, and congruence $\cong$ on segments or angles.

- A **line** $l = \overleftrightarrow{AB}$ extends infinitely in both directions.
- A **segment** is
$$\overline{AB} = \{A, B\} \cup \{C : A * C * B\}$$
with **endpoints** A, B and **interior points** C.
- A **ray** is
$$\overrightarrow{AB} = \overline{AB} \cup \{C : A * B * C\}$$
with **vertex** A.
- A **triangle** is
$$\triangle ABC = \overline{AB} \cup \overline{BC} \cup \overline{CA}$$
where A, B, C are noncollinear. Two triangles are **congruent** if their sides and angles are pairwise congruent.
- $A \neq B \notin l$ have the same **sidedness** with respect to l if $\overline{AB} \cap l = \emptyset$, and otherwise have opposite sidedness.
- An **angle** is
$$\angle BAC = \overrightarrow{AB} \cup \overrightarrow{AC}$$
with **vertex** A and **sides** $\overrightarrow{AB}, \overrightarrow{AC}$.
- l, m **intersect** if $l \cap m \neq \emptyset$, and otherwise are **parallel**.

The **axioms of incidence** are

1. If $A \neq B$, then there is $l \ni A, B$.
2. If $A \neq B$, then there is at most one $l \ni A, B$.
3. If l is a line, then there are $A \neq B \in l$. There are three noncollinear points.

The **axioms of order** are

1. If $A * B * C$, then $A \neq B \neq C \in l$ and $C * B * A$.
2. If $A \neq B$, then there is $A * B * C$.
3. If $A \neq B \neq C \in l$, then exclusively $A * B * C$, $A * C * B$, or $B * A * C$.

4. **Pasch's axiom**: If $\triangle ABC$ is a triangle, $A, B, C \notin l$, and l intersects $\overline{AB}$, then l intersects $\overline{AC}$ or $\overline{BC}$.

The **axioms of congruence** are

1. **Segment transference**: If $A \neq B$ and r is a ray at A', then there is unique $B' \in r$ such that $\overline{AB} \cong \overline{A'B'}$. Further, $\overline{AB} \cong \overline{BA}$.
2. If $\overline{AB} \cong \overline{EF}$ and $\overline{CD} \cong \overline{EF}$, then $\overline{AB} \cong \overline{CD}$.
3. If $A * B * C$, $A' * B' * C'$, $\overline{AB} \cong \overline{A'B'}$, and $\overline{BC} \cong \overline{B'C'}$, then $\overline{AC} \cong \overline{A'C'}$.
4. **Angle transference**: If $\angle BAC$ is an angle and $\overrightarrow{A'B'}$ is a ray, then there is unique $\overrightarrow{A'C'}$ on each side of $\overleftrightarrow{A'B'}$ such that $\angle BAC \cong \angle B'A'C'$.
5. If $\angle ABC \cong \angle GHI$ and $\angle DEF \cong \angle GHI$, $\angle ABC \cong \angle DEF$. Further, $\angle ABC \cong \angle CBA$.
6. **Side-angle-side**: If $\triangle ABC, \triangle A'B'C'$ are triangles, $\overline{AB} \cong \overline{A'B'}$, $\overline{AC} \cong \overline{A'C'}$, and $\angle BAC \cong \angle B'A'C'$, then $\triangle ABC \cong \triangle A'B'C'$.

The **axiom of continuity** states that, if Σ_1, Σ_2 partition l and are pairwise not between each other, then there is unique $\mathcal{O}$ between them:

$$A * \mathcal{O} * B \iff A \in \Sigma_1 \setminus \{\mathcal{O}\} \wedge B \in \Sigma_2 \setminus \{\mathcal{O}\}$$

Playfair's axiom states that, if $P \notin l$, then there is at most one parallel $m \ni P$.

An **incidence geometry** is a model satisfying the axioms of incidence. The **Fano plane** is the seven-point incidence geometry with three lines per point and three points per line. $l \neq m$ intersect at at most one point. If A is a point, then there are $l \neq m \ni A$.

Every segment has an interior point. The **plane separation theorem** states that sidedness is transitive. I is **interior** to $\angle BAC$ if it is on the same side of $\overleftrightarrow{AB}$ as C and the same side of $\overleftrightarrow{AC}$ as B. I is **interior** to $\triangle ABC$ if it is interior to $\angle ABC, \angle BAC, \angle ACB$. Every angle has an interior point. The **crossbar theorem** states that, if I is interior to $\angle BAC$, then $\overrightarrow{AI}$ intersects $\overline{BC}$.

Congruence is an equivalence relation. $\overline{AB} < \overline{CD}$ if $\overline{AB} \cong \overline{CE}$ and $C * E * D$. Angle-side-angle, side-side-side, and side-angle-angle are triangle congruence relations. An isosceles triangle has congruent base angles.

A **circle** is

$$\mathcal{C} = \{A : \overline{OA} \cong \overline{OR}$$

with **center** O and **radius** $\overline{OR}$. P is **inside** $\mathcal{C}$ if $\overline{OP} < \overline{OR}$, and **outside** if $\overline{OP} > \overline{OR}$. The **elementary continuity principle**, or **line–circle continuity principle**, states that, if P is inside and Q is outside, then $\overline{PQ}$ intersects $\mathcal{C}$ at exactly one point. The **circular continuity principle** states that, if $\mathcal{D}$ at O' contains a point inside and a point outside, then $\mathcal{C}, \mathcal{D}$ intersect at exactly two points, which lie on opposite sides of $\overleftrightarrow{OO'}$.

A **chord** is a segment between two points on $\mathcal{C}$. A **diameter** is a chord through O. An **arc** is an interval on $\mathcal{C}$, and is **major** or **minor** by length. If $A, B, P \in \mathcal{C}$, then $\angle AOB = 2\angle APB$ is **central** and $\angle APB$ is **inscribed**. $\mathcal{C}$ is the unique **circumcircle** of any inscribed triangle. If two inscribed triangles share a side, then the angles on the opposite arc are congruent. An inscribed quadrilateral has supplementary opposite angles. **Thales' theorem** states that, if an inscribed triangle has a diameter, then the angle on the opposite arc is right.

l is **tangent** to $\mathcal{C}$ if it intersects exactly once, at T. Equivalently, l is perpendicular to $\overleftrightarrow{OT}$. There are exactly two tangent lines through a given outside point.

1.2 The Measure Axioms

The **length measure axioms** are

1. $\overline{AB}$ has a unique **length** $|AB| \in \mathbb{R}^{\geq}$.
2. $|AB| = |CD| \iff \overline{AB} \cong \overline{CD}$.
3. $|AB| < |CD| \iff \overline{AB} < \overline{CD}$.
4. If $A * B * C$, then $|AB| + |BC| = |AC|$.

The **angle measure axioms**, or **degree measure axioms**, are

1. $\angle ABC$ has a unique **degree measure** $m\angle ABC \in (0, 180)$.
2. $m\angle ABC = m\angle DEF \iff \angle ABC \cong \angle DEF$.
3. $m\angle ABC < m\angle DEF \iff \angle ABC < \angle DEF$.
4. If P is interior to $\angle ABC$, then $m\angle ABP + m\angle PBC = m\angle ABC$.
5. A right angle measures $90°$.

Given $\overline{OP}$, there is a unique length measure scheme such that $|OP| = 1$. The angle measure scheme is unique.

The **area** of a rectangle is the product of the base and height. The **height** of a triangle with respect to a base is the length of the perpendicular to the opposite vertex. The area of a triangle is half the product of the base and height.

Triangles are **similar** $\triangle ABC \sim \triangle XYZ$ if, equivalently,

- $\frac{|AB|}{|XY|} = \frac{|BC|}{|YZ|} = \frac{|CA|}{|ZX|}$;
- they have angle-angle-angle congruence;
- the base of one is parallel to the base of a triangle congruent to the other with congruent opposite angle.

If $\angle ABC$ is acute and $D \in \overrightarrow{BC}$ makes $\angle ADB$ right, then

$$\sin\angle ABC = \frac{|AD|}{|AB|}, \quad \cos\angle ABC = \frac{|BD|}{|AB|}$$

Angles have the same sine or cosine if and only if they are congruent. A **cevian** is a segment from a vertex to the opposite side of a triangle. **Ceva's theorem** states that cevians $\overline{AX}, \overline{BY}, \overline{CZ}$ intersect at a common point if and only if

$$\frac{|BX|}{|XC|}\frac{|CY|}{|YA|}\frac{|AZ|}{|ZB|} = 1$$

If $\overline{AD}, \overline{PQ}$ are chords intersecting at X, then

$$|AX||XD| = |PX||XQ|$$

The **butterfly theorem** states that, if $\overline{PQ}$ is a chord with midpoint M, $\overline{AC}, \overline{BD}$ are chords through M, and X, Y are the respective intersections of $\overline{AD}, \overline{BC}$ with $\overline{PQ}$, then M is the midpoint of $\overline{XY}$.

2 Analytic Geometry

2.1 Cartesian Geometry

Analytic geometry is algebraic and coordinated. **Cartesian geometry** is on the vector space $\mathbb{R}^2$ under the Euclidean norm. If $C = A + B$, then the quadrilateral $OACB$ is a parallelogram.

$$X : \overleftrightarrow{PQ} \to \mathbb{R} : t \mapsto P + t(Q - P) = (1-t)P + tQ$$

is a bijection, and $d(P, X_t) = |t||PQ|$. The medians of a triangle meet at a common point $\frac{2}{3}$ of the way along each median.

$A \neq B \neq C$ have a unique **radian measure**

$$\beta = \measuredangle ABC = \frac{s}{r} \in [0, 2\pi)$$

where s is the counterclockwise arc length from $\overrightarrow{BA}$ to $\overrightarrow{BC}$ on the circle of radius r. If P lies on a circle about the origin and $\theta = \measuredangle IOP$, then

$$\cos\theta = \frac{x}{r}, \quad \sin\theta = \frac{y}{r}, \quad \tan\theta = \frac{y}{x}$$

The **law of sines** states that, if d is the circumcircle diameter, then

$$\frac{\sin\alpha}{a} = \frac{\sin\beta}{b} = \frac{\sin\gamma}{c} = \frac{1}{d}$$

The **law of cosines** states

$$c^2 = a^2 + b^2 - 2ab\cos\gamma$$

A **Euclidean isometry** is

$$f : \mathbb{R}^2 \to \mathbb{R}^2 : |PQ| = |f(P)f(Q)|$$

Two objects are **isometric**, or **congruent**, if there is an isometry mapping one to the other.

$$\angle PQR \cong \angle f(P)f(Q)f(R)$$

Every Euclidean isometry has

$$f(X) = g(X) + C$$

where g is an origin-preserving isometry.

Complex geometry is on the vector space $\mathbb{C}$ under the complex modulus. Common transformations are

- **Translation** by w or $\vec{w}$:

$$z \mapsto z + w, \quad \vec{z} \mapsto \vec{z} + \vec{w}$$

- **Scaling** by λ:

$$z \mapsto \lambda z, \quad \vec{z} \mapsto \lambda \vec{z}$$

- **Rotation** by θ:

$$z \mapsto e^{i\theta} z, \quad \vec{z} \mapsto \begin{bmatrix} \cos\theta & -\sin\theta \\ \sin\theta & \cos\theta \end{bmatrix} \vec{z}$$

- **Reflection** across $\frac{\theta}{2}$:

$$z \mapsto e^{i\theta} \bar{z}, \quad \vec{z} \mapsto \begin{bmatrix} \cos\theta & \sin\theta \\ \sin\theta & -\cos\theta \end{bmatrix} \vec{z}$$

2.2 Birkhoff's Axioms

Birkhoff geometry is an axiomatization of Cartesian geometry. The axioms are

- **Euclidean**: If $A \neq B$, then there is $l \ni A, B$.
- **Ruler**: There is a bijection t between points in l and $\mathbb{R}$ such that $|t_A - t_B| = d(A, B)$.
- **Protractor**: There is a continuous bijection θ between rays from O and $[0, 2\pi)$ such that $\measuredangle AOB \cong \beta - \alpha \pmod{2\pi}$.
- **SAS similarity**.

$A * B * C$ if $d(A, B) + d(B, C) = d(A, C)$.

3 Absolute Geometry

3.1 Absolute Geometry

Absolute geometry, or **neutral geometry**, is Hilbert plane geometry without Playfair's axiom. The **Saccheri–Legendre theorem** states $\Sigma_\triangle \leq 180°$. A **Saccheri quadrilateral** $ABCD$ has **base** $\overline{AB}$ and **summit** $\overline{CD}$, with $\overline{AD} \cong \overline{BC}$ and $\measuredangle DAB = \measuredangle CBA = 90°$. A **Lambert quadrilateral** has three right angles. Bisecting the base of a Saccheri quadrilateral gives two congruent Lambert quadrilaterals. The summit angles of a Saccheri quadrilateral are congruent and acute. In Hilbert plane geometry, these quadrilaterals are rectangles.

The **area axioms** are

1. Two shapes have the same area if and only if they can be divided into finitely many pairwise-congruent triangles.

2. The area of a triangle is positive.

3. The area of a union of disjoint shapes is the sum of the shapes' areas.

The **angle defect** of $\triangle$ is $\pi - \Sigma_\triangle$. The sum of angle defects of two subtriangles is the angle defect of the supertriangle. If two triangles have the same area, then they have the same angle sum.

If $\triangle ABC$ is a triangle, then bisecting $\overline{AB}$ at E and $\overline{AC}$ at F and dropping perpendiculars from B and C to $\overleftrightarrow{EF}$ at H and J gives a Saccheri quadrilateral $HICB$. If $HICB$ is a Saccheri quadrilateral with summit $\overline{BC}$, then letting $\overleftrightarrow{HI}$ bisect $\overline{AB}$ at E makes $\overleftrightarrow{HI}$ bisect $\overline{AC}$. These triangle and quadrilateral have equal area, and the summit angle sum equals the triangle angle sum.

3.2 Hyperbolic Geometry

The **Bolyai–Lobachevsky postulate**, or **hyperbolic postulate**, states that there are at least two parallels $m \ni P \notin l$. **Hyperbolic geometry** is absolute geometry with the Bolyai–Lobachevsky postulate. The **theorem egregium** states that a hyperbolic model cannot have both Hilbert lines and Hilbert angles.

The **Poincaré disk** is the interior of the unit Hilbert circle. An **omega-point** is a boundary point. A **hyperbolic line** is a Hilbert-circular arc meeting omega points at right angles:

- a diameter
$$dx = cy$$

- an arc on a Hilbert circle with center (a, b) and radius $\sqrt{a^2 + b^2 - 1}$
$$x^2 + y^2 - 2ax - 2by + 1 = 0$$

The **hyperbolic distance** is

$$d(P,Q) = \cosh^{-1}\left(1 + \frac{2|PQ|^2}{(1-|P|^2)\,(1-|Q|^2)}\right)$$

The **angle** between two rays is the Hilbert angle between their tangent lines.

$$d(O,P) = \cosh^{-1}\frac{1+|P|^2}{1-|P|^2} = \ln\frac{1+|P|}{1-|P|}$$

If $P, I \in l$, then

$$d_P : \overrightarrow{PI} \to (0,\infty) : Q \mapsto d(P,Q)$$

is bijective and differentiable.

Some hyperbolic models are as follows:

- The Poincaré disk.
- The **Klein disk**: the Poincaré disk where lines are Hilbert chords and

$$d(P,Q) = \frac{1}{2}\left|\ln\frac{|P\Theta||Q\Omega|}{|P\Omega||Q\Theta|}\right|$$

  Then a perpendicular is a Hilbert chord whose extension passes through the intersection of the tangents at Ω and Θ.

- The **Poincaré half-plane**: $y > 0$ where lines are

$$x = k, \quad (x-a)^2 + y^2 = r^2$$

- The **hyperboloid**: $x^2 + y^2 = z^2 - 1$ where $z \geq 1$, lines are intersections with planes through O, and

$$d(P,Q) = \cosh^{-1}(cz - ax - by)$$

There is a unique perpendicular $\overleftrightarrow{PQ} \ni P \notin l$. Then $m \ni P$ is a **limiting parallel**, or **asymptotic parallel**, if there is a parallel $n \ni P$ such that

- $\overrightarrow{PI}$ intersects l if and only if $m * \overrightarrow{PI} * n$, and
- m, n make angle $\mu < 90°$ with $\overrightarrow{PQ}$, the **angle of parallelism**

and is otherwise an **ultraparallel**. The **fundamental theorem of parallels** states that there are exactly two limiting parallels through P. Finally,

$$\cosh d(P,Q) = \csc\mu, \quad \tan\frac{\mu}{2} = e^{-d(P,Q)}$$

There are no quadrilaterals. There are no triangles of angle sum 180°. Angle-angle-angle congruence holds.

An **omega-triangle**, or **ideal-triangle**, is restricted from the closure of the Poincaré disk and has at least two sides that are limiting parallels of each other. Equivalently, at least one vertex is an omega-point. The **exterior angle theorem** states that, if $\triangle DE\Omega$ has exactly one omega-point and $D * E * F$, then $\angle FE\Omega > \angle ED\Omega$. Then angle-angle and side-angle on the Poincaré disk are congruence relations.

If two triangles have a congruent side and equal angle sum, then they have equal area. In fact, they need not have a congruent side.

$$\pi - \Sigma_\triangle = A_\triangle$$

The **complex Poincaré disk** is $D = \{z \in \mathbb{C} : |z| < 1\}$.

$$d(z, w) = \left| \ln \frac{|z - \Omega||w - \Theta|}{|z - \Theta||w - \Omega|} \right|$$

A **Möbius transformation**, or **fractional-linear transformation**, has

$$f(z) = \frac{az + b}{cz + d}$$

with $ad - bc \neq 0$, and preserves lines, angles, and circles.

$$f^{-1}(z) = \frac{dz - b}{-cz + a}$$

$$\frac{(f(z_1) - f(z_2))(f(z_3) - f(z_4))}{(f(z_2) - f(z_3))(f(z_4) - f(z_1))} = \frac{(z_1 - z_2)(z_3 - z_4)}{(z_2 - z_3)(z_4 - z_1)}$$

The isometries are

- **Translation** by $-\alpha$:
$$z \mapsto \frac{\alpha - z}{\bar{\alpha}z - 1}$$
- **Rotation** by θ:
$$z \mapsto e^{i\theta} z$$
- **Reflection** across $\frac{\theta}{2}$:
$$z \mapsto e^{i\theta} \bar{z}$$

On a right triangle,

$$\sin\beta = \frac{\sinh b}{\sinh c}, \quad \cos\beta = \frac{\tanh a}{\tanh c}, \quad \tan\beta = \frac{\tanh b}{\sinh a}$$

The **hyperbolic Pythagorean theorem** states

$$\cosh c = \cosh a \cosh b$$

The **hyperbolic law of sines** states

$$\frac{\sinh a}{\sin \alpha} = \frac{\sinh b}{\sin \beta} = \frac{\sinh c}{\sin \gamma}$$

The **first hyperbolic law of cosines** states

$$\cosh c = \cosh a \cosh b - \sinh a \sinh b \cos \gamma$$

The **second hyperbolic law of cosines** states

$$\cos \gamma = \sin \alpha \sin \beta \cosh c - \cos \alpha \cos \beta$$

A **geodesic** is a path of minimal length. In hyperbolic geometry, the geodesics are the hyperbolic lines.

A Axiomatic Systems

An **axiomatic system** has

- **undefined terms**: concepts accepted without definition;
- **axioms**: logical statements about undefined terms accepted without proof;
- **defined terms**: concepts defined in terms of other objects;
- **theorems**: logical statements following from other objects.

A **proof** is a logical argument for a theorem. A **model** is a set of definitions for undefined terms such that all axioms are true. An **abstract model** depends on another axiomatic system.

An axiomatic system is **consistent** if there are no contradictions, **independent** if the axioms are independent of each other, and **complete** if every proposition is decidable. **Gödel's incompleteness theorems** state that, if a system contains the natural numbers, then

1. the system is incomplete, and
2. that the system is consistent is undecidable.

B The Euclidean System

Euclidean plane geometry is a system on points, lines, circles, and angle relations. It contains flaws that prevent it from being axiomatic, but all the results here also hold for Hilbert's axioms. In this system, an **axiom** is an abstract statement without proof:

1. Congruence is transitive.
2. Congruencies can be added to each other.
3. Congruencies can be subtracted from each other.
4. Congruence is reflexive.
5. Every set has a partial order.

A **postulate** is a geometric statement without proof:

1. Two points give a segment.
2. A segment may be extended infinitely in either direction.
3. Two points give a circle.
4. Right angles are equal.

5. If a line intersects two others and the internal angles on one side sum to less than a straight angle, then the two lines intersect on that side.

Postulates 1–3 are the **ruler-and-compass constructions**. Postulate 5 is the **parallel postulate**, and is equivalent to Playfair's postulate.

Given a segment, we can construct an equilateral triangle. Triangles with side-angle-side congruence are congruent. An isosceles triangle has congruent base angles. We can bisect an angle or segment. The **vertical angle theorem** states that, if two lines intersect, then the opposite angles at the intersection are congruent.

The **exterior angle theorem** states that, if one side of a triangle is extended, then the created exterior angle is congruent to the sum of the opposite interior angles. If a line intersects two lines, then the alternate angles are congruent if and only if the two lines are parallel. Equivalently, the line makes the same angle with both lines.

The Pythagorean theorem states that a triangle is right if and only if the square of some side is congruent to the sum of the squares of the other sides.

C Fractals

An object is **self similar** if it is partitioned by N copies of itself, each scaled by the **magnification factor** r. The **fractal dimension** is

$$D = \log_{\frac{1}{r}} N = \frac{\log N}{\log \frac{1}{r}} = -\frac{\log N}{\log r}$$

A **contraction mapping** with **scale factor** $c \in [0, 1)$ is

$$S : \mathbb{R}^m \to \mathbb{R}^m : |S(x) - S(y)| \leq c|x - y|$$

L is a **fixed point** if $S(L) = L$. The **Banach fixed point theorem** states that there is a unique fixed point, and, if $x_{n+1} = S(x_n)$, then $\lim x_n = L$. The **iterated function systems theorem** states that, if $\mathcal{H}$ is a complete metric space and $S(K) = \bigcup_{i=1}^{n} S_i(K)$, then S is a contraction mapping with contraction factor $c = \max\{c_1, \ldots, c_n\}$, and the **fractal**, or **attractor**, $F = \lim S^k(K_0)$ is fixed under S. Further, usually, D is the unique number such that

$$\sum_{i=1}^{n} c_i^D = 1$$

If $K \subseteq \mathbb{R}^m$ is compact, then the **minimal ε-covering number** is

$$\mathcal{N}(K, \varepsilon) = \min \left\{ M : K \subseteq \bigcup_{n=1}^{M} \overline{B_\varepsilon(x_n)} \right\}$$

The **fractal dimension** is

$$D = \lim_{\varepsilon \to 0} \frac{\log \mathcal{N}(K, \varepsilon)}{\log \frac{1}{\varepsilon}}$$

The **box-counting theorem** states that, if $\mathcal{N}_n(K)$ is the number of squares of side length $\frac{1}{2^n}$ tiling $\mathbb{R}^m$ and intersecting K, then

$$D = \lim_{n \to \infty} \frac{\log \mathcal{N}_n(K)}{\log 2^n}$$